Data Science

TD 4

1. Optimal Filtering and the Wiener Filter

A geophysicist measures a seismic signal

$$y(t) = x(t) + n(t)$$

where:

- $x(t)$ is the stationary seismic signal of interest. It is centered ($\mu_x = 0$) with autocorrelation function $\gamma_x(\tau) = \sigma_x^2 \, e^{-\alpha|\tau|}$
- $n(t)$ is stationary measurement noise, it is centered and independent of $x(t)$
- the signal and the noise are uncorrelated: $\mathbb{E}\{x(t)\, n(t')\} = 0$

The goal is to design an **optimal linear filter** $h(t)$ that estimates $x(t)$ from the noisy measurements $y(t)$. We seek an estimate $x^+(t)$ that minimizes the **mean square error (MSE)**:

$$\text{MSE} = \int_{\mathbb{R}} \mathbb{E}\Big\{ |x(t) - x^+(t)|^2 \Big\} \, \mathrm{d}t$$

1.1. Computing power spectral densities

Q. 1 The noise is assumed independent, white, and identically distributed with variance σ_n^2.

1-a) Write its autocorrelation function.

1-b) Compute the autocorrelation $\gamma_{y(\tau)}$ of the measured signal y.

1-c) Compute the cross-correlation $\gamma_{x,y}(\tau) = \int_{\mathbb{R}} E\{x(t)\, y(t+\tau)\} \, \mathrm{d}t$ between the true signal and the measurements.

Q. 2 Compute the power spectral densities of:

- the true signal $S_x(\nu)$
- the noise $S_n(\nu)$
- the measurements $S_y(\nu)$
- the cross power spectral density between x and y: $S_{x,y}(\nu)$

1.2. Derivation of the Wiener filter

The Wiener filter is the filter with transfer function $\hat{h}$ that minimizes the Mean Square Error. From Parseval-Plancherel we can write:

$$\text{MSE} = \int_{\mathbb{R}} \mathbb{E}\Big\{ \big|\hat{x}(\nu) - \hat{h}(\nu)\, \hat{y}(\nu)\big|^2 \Big\} \, \mathrm{d}\nu$$

Q. 3 3-a) Show that

$$\text{MSE} = \int_{\mathbb{R}} S_x(\nu) - 2\,\Re\Big(\hat{h}(\nu)\, S_{x,y}(\nu)\Big) + \left|\hat{h}(\nu)\right|^2 S_y(\nu)\,\mathrm{d}\nu$$

3-b) To minimize the MSE, take the derivative with respect to $\hat{h}(\nu)$ and set it to zero. Show that the optimal filter (Wiener filter) satisfies:

$$\hat{h}(\nu) = \frac{S_x(\nu)}{S_x(\nu) + S_n(\nu)}, \quad \forall \nu \in \mathbb{R}$$

1.3. Application

Q. 4 4-a) Write the Wiener filter expression using the densities computed in question 2:

4-b) Analyze the behavior of the Wiener filter:
- What happens at low frequencies?
- What happens at high frequencies?
- How does the filter depend on the signal-to-noise ratio: $\text{SNR} = \frac{\sigma_x^2}{\sigma_n^2}$?

4-c) Sketch (qualitatively) the magnitude $\left|\hat{h}(\nu)\right|^2$

2. Spatial Interpolation and Kriging

A geophysicist has measured a physical property (e.g., porosity, seismic velocity) at N discrete locations $\boldsymbol{x} = [x_1, x_2, ..., x_N]^T$ in a geological formation. The measurements are $Z(x_i) = z_i$ (*i.e.* the vector $\boldsymbol{z} = [z_1, z_2, ..., z_N]^T$).

The goal is to estimate the value $\tilde{Z}(x_0)$ at an unmeasured location x_0 using a linear combination of the observations:

$$\tilde{Z}(x_0) = \sum_{i=1}^{N} \lambda_i Z(x_i)$$

2.1. Expressing the covariance structure

We model $Z(x)$ as a stationary random field (spatial random signal) with zero mean $\mathbb{E}\{Z(x)\} = 0$ and covariance function

$$\gamma(h) = \mathbb{E}\big\{Z(x)\overline{Z}(x+h)\big\} = \sigma^2 e^{-\frac{|h|}{a}}$$

where a is the correlation length (range) and σ^2 a constant.

Q. 5 5-a) What is the covariance $\gamma(0)$ at zero distance? Interpret this value.

5-b) How does the covariance drop at a distance a?

2.2. Optimal weights

We want to find weights $\boldsymbol{\lambda} = [\lambda_1, \lambda_2, ..., \lambda_N]$ that minimize the mean square prediction error:

$$\text{MSE} = \mathbb{E}\Big\{\left|\tilde{Z}(x_0) - Z(x_0)\right|^2\Big\}$$

Q. 6 6-a) Show that the prediction error can be written as:

$$\text{MSE} = \gamma(0) - 2\sum_{i=1}^{N} \lambda_i \gamma(x_0 - x_1) + \sum_{i=1}^{N}\sum_{j=1}^{N} \lambda_i \lambda_j \gamma\big(x_i - x_j\big)$$

6-b) Find the system of linear equations in λ_i that minimizes the MSE by setting the derivative with respect to λ_i to zero.

6-c) Rewrite it in a matrix form leading to the simple kriging equations:

$$\boldsymbol{\lambda} = \boldsymbol{C}^{-1} \cdot \boldsymbol{c}$$

$$\tilde{Z}(x_0) = \boldsymbol{z}^T \cdot \boldsymbol{C}^{-1} \cdot \boldsymbol{c} = \sum_{i=1}^{N} \lambda_i Z(x_i)$$

2.3. Properties

Q. 7 Estimate the bias of the estimator $\tilde{Z}(x_0)$, that is $\text{Bias} = \mathbb{E}\big\{\tilde{Z}(x_0) - Z(x_0)\big\}$

Q. 8 Estimate the variance of the estimator $\text{Var}\big(\tilde{Z}(x_0)\big) = \mathbb{E}\Big\{\big|\tilde{Z}(x_0) - Z(x_0)\big|^2\Big\}$