Data Science

TD 3

1. Sampling

Q. 1 Fourier transform of the Dirac comb:

(a) Use the properties of Fourier transform to compute the Fourier transform of

$$Ш(t) = \sum_{n=-\infty}^{+\infty} \delta(t-n)$$

Solution:

Using the property of the Fourier transform of a shifted delta function, we have:

$$\mathcal{F}\{\delta(t-n)\}(\nu) = e^{-j\,2\pi\,\nu\,n}$$

Therefore, the Fourier transform of the Dirac comb is:

$$\hat{Ш}(\nu) = \sum_{n=-\infty}^{+\infty} e^{-j\,2\pi\,\nu\,n}$$

This sum is zero everywhere except at frequencies where ν is an integer, due to the periodicity of the complex exponential function. Specifically, the sum converges to a series of delta functions located at integer frequencies.

Thus, we can express the Fourier transform of the Dirac comb as:

$$\begin{aligned}\hat{Ш}(\nu) &= \sum_{k=-\infty}^{+\infty} \delta(\nu-k)\\ &= Ш(\nu)\end{aligned}$$

This shows that the Fourier transform of a Dirac comb is itself. The Dirac comb is an eigenfunction of the Fourier transform with unit eigenvalue.

(b) Compute the Fourier transform the following Dirac comb (we define $\nu_0 = \frac{1}{T}$):

$$Ш_T(t) = \sum_{n=-\infty}^{+\infty} \delta(t - T\,n)$$

Solution:

Using the scaling property of the Fourier transform, we have:

$$\widehat{\text{Ш}_T}(\nu) = \frac{1}{T}\sum_{k=-\infty}^{+\infty}\delta\left(\nu - \frac{k}{T}\right)$$
$$= \nu_0 \sum_{k=-\infty}^{+\infty}\delta(\nu - k\nu_0)$$
$$= \nu_0 \text{Ш}_{\nu_0}(\nu)$$

This result indicates that the Fourier transform of a Dirac comb with period T is another Dirac comb in the frequency domain, scaled by a factor of $\frac{1}{T}$ and with spikes located at integer multiples of $\nu_0 = \frac{1}{T}$.

Q. 2 Consider the following signal:

$$x(t) = \sin(2\pi\,\nu_x\,t + \phi)$$

Compute its Fourier transform $\hat{x}$ and plot $|\hat{x}|$:

Solution:

Using Euler's formula, we can express the sine function in terms of complex exponentials:

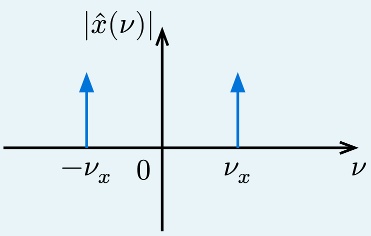
$$\sin(2\pi\,\nu_x t + \phi) = \frac{e^{j(2\pi\,\nu_x t+\phi)} - e^{-j(2\pi\,\nu_x t+\phi)}}{2j}$$
$$= \frac{1}{2}e^{j\frac{\pi}{2}}\left(e^{j\phi}e^{j2\pi\,\nu_x t} - e^{-j\phi}e^{-j2\pi\,\nu_x t}\right)$$

The Fourier transform of $e^{j2\pi\,\nu_x t}$ is given by:

$$\mathcal{F}\left\{e^{j2\pi\,\nu_x t}\right\}(\nu) = \delta(\nu - \nu_x)$$

Using linearity of the Fourier transform, we obtain

$$\mathcal{F}\{x(t)\}(\nu) = \frac{1}{2}e^{-j\frac{\pi}{2}}\left(e^{j\phi}\delta(\nu - \nu_x) - e^{-j\phi}\delta(\nu + \nu_x)\right)$$
$$= \frac{e^{j(\phi - \frac{\pi}{2})}}{2}\delta(\nu - \nu_x) - \frac{e^{-j(\phi - \frac{\pi}{2})}}{2}\delta(\nu + \nu_x)$$



Q. 3 We defined:

$$y(t) = x(t)\,\text{Ш}_T(t)$$

Compute the Fourier transform $\hat{y}$ and plot y and $|\hat{y}|$.

Solution:

Using the property of the Fourier transform of a product of two functions, we have:

$$\begin{aligned}\mathcal{F}\{y(t)\}(\nu) &= \mathcal{F}\{x(t)\,\text{Ш}_T(t)\}(\nu)\\ &= (\hat{x} * \mathcal{F}\{\text{Ш}_T(t)\})(\nu)\\ &= \left(\frac{e^{j(\phi-\frac{\pi}{2})}}{2}\delta(\nu-\nu_x) - \frac{e^{-j(\phi-\frac{\pi}{2})}}{2}\delta(\nu+\nu_x)\right) * \left(\frac{1}{T}\sum_{k=-\infty}^{+\infty}\delta(\nu-k\nu_0)\right)\\ &= \nu_0 \sum_{k=-\infty}^{+\infty}\left[\frac{e^{j(\phi-\frac{\pi}{2})}}{2}\delta(\nu-\nu_x-k\nu_0) - \frac{e^{-j(\phi-\frac{\pi}{2})}}{2}\delta(\nu+\nu_x-k\nu_0)\right]\\ &= \text{Ш}_{\nu_0}(\nu-\nu_s) + \text{Ш}_{\nu_0}(\nu+\nu_s)\end{aligned}$$

This result indicates that the Fourier transform of the modulated signal $y(t)$ consists of shifted replicas of the original spectrum $\hat{x}(\nu)$, located at integer multiples of the sampling frequency $\nu_0 = \frac{1}{T}$.

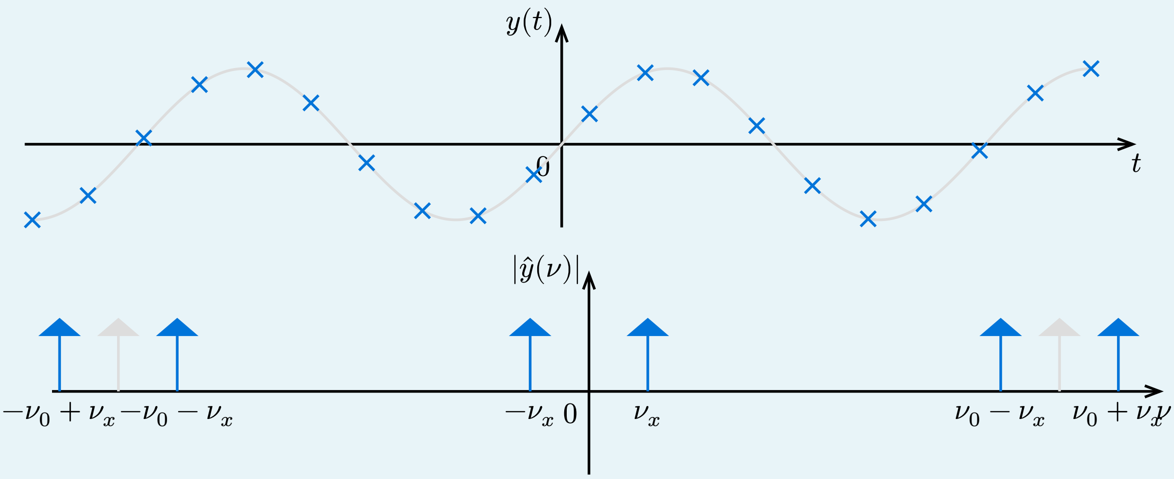


Figure 1: Sampling a periodic signal of period 4 with sampling period 0.5

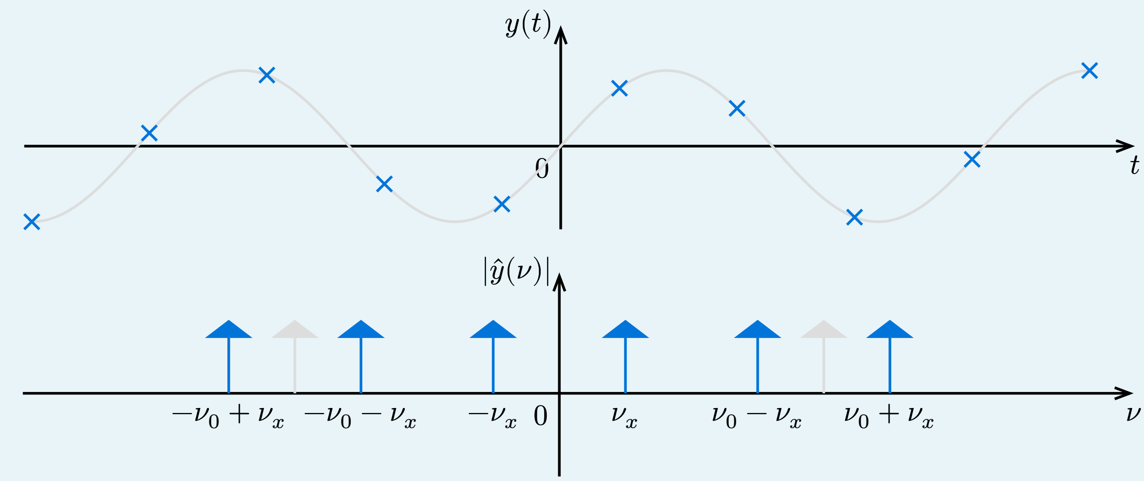


Figure 2: Sampling periodic signal with period 4 with sampling period 1

Q. 4 Redraw the plot when x is a band-limited signal of spectral width B_s. ($i.e.$ $\hat{x}(\nu) = 0$ if $|\nu| > \frac{B_s}{2}$)

Q. 5 What will happen if $\nu_e < B_s$?

Solution:

If the sampling frequency ν_e is less than twice the highest frequency component ν_x of the signal (i.e., $\nu_e < 2\nu_x$), aliasing will occur. Aliasing is a phenomenon where higher frequency components of the signal are indistinguishably mapped to lower frequencies in the sampled signal.

In the frequency domain, this means that the shifted replicas of the original spectrum $\hat{x}(\nu)$ will overlap. Specifically, the replicas located at $k\nu_0 \pm \nu_x$ for different integer values of k will interfere, causing distortion in the reconstructed signal.

As a result, when attempting to reconstruct the original signal from its samples, it will not be possible to accurately recover the original frequencies, leading to a loss of information and potential distortion in the signal representation.

The sampling of the signal y is performed by defining $y[k] = y(kT)$.

2. Reconstructing sampled data

From the samples $y[k]$, we can reconstruct the continuous signal $y(t)$:

$$y(t) = \begin{cases} y[\frac{t}{T}] & \text{if } \frac{t}{T} \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

From $y(t)$, we obtain a reconstruction $z(t)$ of the original signal $x(t)$ by applying a filter with impulse response h:

$$z(t) = (h * y)(t)$$

Q. 6 Knowing that x is a band-limited signal of spectral width B_s, show that the optimal filter that minimizes the reconstruction error $E = \int_{-\infty}^{+\infty} |x(t) - z(t)|^2 \,\mathrm{d}t$ is a cardinal sine.

Solution:

$$\begin{aligned} E &= \int_{-\infty}^{+\infty} |x(t) - z(t)|^2 \,\mathrm{d}t \\ &= \int_{-\infty}^{+\infty} |\hat{x}(\nu) - \hat{h}(\nu)\,\hat{y}(\nu)|^2 \,\mathrm{d}\nu \end{aligned}$$

The signal $\hat{x}(\nu)$ is non-zero only on the interval $\left[-\frac{B_s}{2}, +\frac{B_s}{2}\right]$.

$$E = \int_{-\infty}^{+\frac{B_s}{2}} |\,\hat{h}(\nu)\,\hat{y}(\nu)|^2 \,\mathrm{d}\nu + \int_{-\frac{B_s}{2}}^{+\frac{B_s}{2}} |\hat{x}(\nu) - \hat{h}(\nu)\,\hat{y}(\nu)|^2 \,\mathrm{d}\nu + \int_{+\frac{B_s}{2}}^{+\infty} |\,\hat{h}(\nu)\,\hat{y}(\nu)|^2 \,\mathrm{d}\nu$$

As consequence the error E is minimized when $\hat{h}(\nu)$ is zero outside the interval $\left[-\frac{B_s}{2}, +\frac{B_s}{2}\right]$ and $\hat{x}(\nu) = \hat{h}(\nu)\,\hat{y}(\nu)$ inside.

If $\nu_0 > B_s$, only a single replica of $\hat{x}$ is present in the interval $\left[-\frac{B_s}{2}, +\frac{B_s}{2}\right]$ and $\hat{y}(\nu) = \nu_0\hat{x}(\nu)$.

$$\begin{aligned} \hat{h}(\nu) &= \begin{cases} 1 & \text{if } |\nu| \leq \frac{B_s}{2}, \\ 0 & \text{otherwise.} \end{cases} \\ &= \mathrm{rect}_{B_s}(\nu) \end{aligned}$$

The inverse Fourier transform of the rectangle function is the cardinal sine:

$$h(t) = \mathcal{F}^{-1}\left\{\hat{h}(\nu)\right\}(t)$$
$$= B_s \,\mathrm{sinc}(B_s\, t)$$

(a) Is this filter causal? What are the issues with such filter?

Solution:

This filter is not causal as its impulse response $h(t)$ is non-zero for negative t. A causal filter must have an impulse response that is zero for all negative time values, meaning it cannot respond to future inputs.

The issues with this filter include:

- Delay: The filter would require knowledge of future input values, preventing real-time processing.
- Infinite length: The sinc function extends infinitely in both directions, making it impractical to implement in real-world systems where finite-length filters are required.

Q. 7 We design a filter with the following impulse response:

$$g(t) = \frac{1}{\tau} u(t) e^{-\frac{t}{\tau}}$$

(a) Compute the transfer function of this filter g.

Solution:

The transfer function $\hat{g}(\nu)$ of the filter can be computed using the Fourier transform of its impulse response $g(t)$:

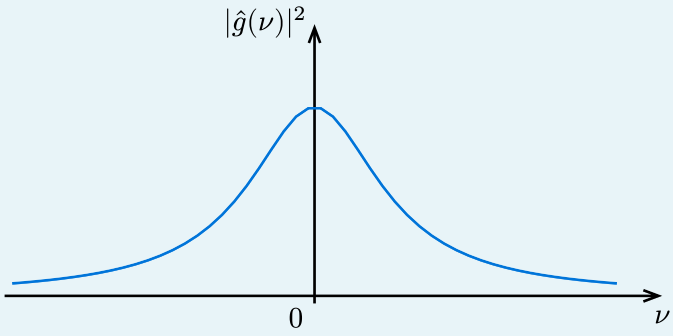
$$\begin{aligned}
\hat{g}(\nu) &= \int_0^{+\infty} \frac{1}{\tau} e^{-\frac{t}{\tau}} e^{-j\,2\pi\,\nu\,t}\,\mathrm{d}t \\
&= \frac{1}{\tau} \int_0^{+\infty} e^{-t\left(\frac{1}{\tau}+j\,2\pi\,\nu\right)}\,\mathrm{d}t \\
&= \frac{1}{\tau}\left[-\frac{1}{\frac{1}{\tau}+j\,2\pi\,\nu} e^{-t\left(\frac{1}{\tau}+j\,2\pi\,\nu\right)}\right]_0^{+\infty} \\
&= \frac{1}{1+j\,2\pi\,\nu\,\tau}
\end{aligned}$$

(b) Compute its squared modulus $|\hat{g}|^2$ and its phase $\angle(\hat{g})$; plot both quantities:

Solution:

The square modulus of the transfer function $\hat{g}(\nu)$ is given by:

$$|\hat{g}(\nu)|^2 = \frac{1}{|1 + j\,2\pi\,\nu\,\tau|^2}$$
$$= \frac{1}{1 + (2\pi\,\nu\,\tau)^2}$$



(c) What is the cut-off frequency ν_c of g:

$$|\hat{g}(\nu_c)|^2 = \frac{1}{2}$$

Solution:

The cut-off frequency ν_c of the filter is defined as the frequency at which the square modulus of the transfer function drops to half its maximum value.

Solving for ν_c, we get:

$$\frac{1}{1 + (2\pi\,\nu_c\,\tau)^2} = \frac{1}{2}$$
$$\Rightarrow (2\pi\,\nu_c\,\tau)^2 = 1$$
$$\Rightarrow \nu_c = \frac{1}{2\pi\,\tau}$$

Therefore, the cut-off frequency of the filter is $\nu_c = \frac{1}{2\pi\,\tau}$.

(d) What could be the time constant τ of this filter to reconstruct x from y?

Solution:

To effectively reconstruct the original signal x from the sampled signal y, the time constant τ of the filter should be chosen such that the cut-off frequency ν_c is greater than or equal to half the bandwidth of the original signal x but smaller than the sampling frequency ν_e to avoid aliasing.

Given that x is band-limited with spectral width B_s, we require:

$$\frac{B_s}{2} < \nu_c < \nu_e$$

$$\Rightarrow \frac{B_s}{2} < \frac{1}{2\pi\,\tau} < \nu_e$$

$$\Rightarrow \frac{1}{2\pi\,\nu_e} < \tau < \frac{1}{\pi\,B_s}$$