Data Science

TD 2

Q. 1 Consider the following signal:

$$x(t) = 1 + \sin(4\pi t) + 2\cos(4\pi t) + \cos\left(6\pi t + \frac{\pi}{4}\right)$$

(a) What is the fundamental pulsation ω_0?

Solution:

The fundamental pulsation is $\omega_0 = 2\pi$.

(b) Compute its Fourier series:

Solution:

Expand $x(t)$ in term of complex exponentials using Euler

$$\begin{aligned} x(t) &= 1 + \frac{e^{j2\omega_0 t} - e^{-j2\omega_0 t}}{2j} + 2\frac{e^{j2\omega_0 t} + e^{-j2\omega_0 t}}{2} + \frac{1}{2}\left(e^{j(3\omega_0 t + \frac{\pi}{4})} + e^{-j(3\omega_0 t + \frac{\pi}{4})}\right) \\ &= 1 + \left(\frac{1}{2j} + 1\right)e^{j2\omega_0 t} + \left(-\frac{1}{2j} + 1\right)e^{-j2\omega_0 t} + \frac{1}{2}e^{j\frac{\pi}{4}}e^{j3\omega_0 t} + \frac{1}{2}e^{-j\frac{\pi}{4}}e^{-j3\omega_0 t} \\ &= 1 + \left(\frac{1}{2j} + 1\right)e^{j2\omega_0 t} + \left(-\frac{1}{2j} + 1\right)e^{-j2\omega_0 t} + \frac{\sqrt{2}}{4}(1-j)e^{j3\omega_0 t} + \frac{\sqrt{2}}{4}(1+j)e^{-j3\omega_0 t} \end{aligned}$$

The Fourier series coefficients are:

$$\begin{aligned} \hat{x}[0] &= 1 \\ \hat{x}[2] &= 1 - \frac{j}{2} \\ \hat{x}[-2] &= 1 + \frac{j}{2} \\ \hat{x}[3] &= \frac{\sqrt{2}}{4}(1+j) \\ \hat{x}[-3] &= \frac{\sqrt{2}}{4}(1-j) \\ \hat{x}[k] &= 0 \text{ for all other k} \end{aligned}$$

Q. 2 Fourier Series of a periodic square wave

(a) Compute the Fourier series of the periodic signal of period T (taking $\omega_0 = \frac{2\pi}{T}$):

$$x(t) = \begin{cases} 1 & \text{if } |t| \leq L \\ 0 & \text{otherwise} \end{cases}$$

Solution:

$\hat{x}[0] = \frac{1}{T}\int_{-L}^{+L} \mathrm{d}t = \frac{2L}{T}$

For $k \neq 0$:

$$\begin{aligned}
\hat{x}[k] &= \frac{1}{T}\int_{-L}^{+L} e^{-jk\omega_0 t}\,\mathrm{d}t \\
&= \frac{1}{T}\left[-\frac{1}{jk\omega_0}e^{-jk\omega_0 t}\right]_{-L}^{L} \\
&= \frac{1}{k\omega_0 T}\left(\frac{e^{jk\omega_0 L} - e^{-jk\omega_0 L}}{j}\right) \\
&= 2\frac{\sin(k\omega_0 L)}{k\omega_0 T} \\
&= \frac{\sin(k\omega_0 L)}{k\pi}
\end{aligned}$$

(b) Compute the Fourier series of the Dirac comb of period T:

$$y(t) = \sum_{k=-\infty}^{+\infty} \delta(t - kT)$$

Solution:

The Fourier series coefficients are given by:

$$\hat{y}[k] = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-jk\omega_0 t}\,\mathrm{d}t$$

Evaluating the integral, we find that all coefficients are equal to $\frac{1}{T}$.

(c) Compute the Fourier series of:

$$z(t) = y(t + T_1) - y(t - T_1)$$

Solution:

Using linearity and time-shifting properties of the Fourier series, we find that the coefficients are:

$$\begin{aligned}
\hat{z}[k] &= \hat{y}[k]\left(e^{jkT_1\omega_0} - e^{-jkT_1\omega_0}\right) \\
&= \hat{y}[k]\,(2j\sin(k\,\omega_0\,T_1)) \\
&= \frac{2j}{T}\sin(k\,\omega_0\,T_1)
\end{aligned}$$

(d) Derive the solution of question (a) from the answer of question (c)

Solution:

We can notice $z(t)$ is the derivative of $x(t)$, we can use the differentiation property

$$\hat{z}[k] = jk\omega_0\hat{x}[k]$$

Thus,

$$\begin{aligned}\hat{x}[k] &= \frac{\hat{z}[k]}{jk\omega_0} \quad \forall k \neq 0 \\ &= \frac{2\sin(k\,\omega_0\,T_1)}{T\,k\,\omega_0}\end{aligned}$$

When $T_1 = L$, it is the same result as in question (a) if we take the mean $\hat{x}[0] = 2\frac{T_1}{T}$.

(e) How many terms are needed to capture 95 % of the signal $x(t)$ power as a function of $\alpha = 2\frac{L}{T}$?

Solution:

The power of the signal is :

$$\begin{aligned}P &= \frac{1}{T}\int_T |x(t)|^2\,\mathrm{d}t \\ &= \frac{1}{T}\int_{-L}^{+L} 1^2\,\mathrm{d}t \\ &= \frac{2L}{T} \\ &= \alpha\end{aligned}$$

The power of the signal is also given by Parseval's theorem:

$$P = \sum_{k=-\infty}^{+\infty} |\hat{x}[k]|^2$$

The power P_K of signal approximated by the $2K+1$ first terms:

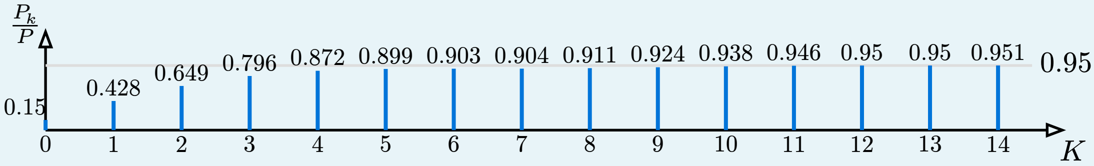
$$\begin{aligned}P_K &= |\hat{x}[0]|^2 + \sum_{k=1}^{K} |\hat{x}[k]|^2 + |\hat{x}[-k]|^2 \\ &= \left(\frac{2L}{T}\right)^2 + \sum_{k=1}^{K}\left(\frac{\sin(k\omega_0 L)}{k\pi}\right)^2 + \left(\frac{\sin(-k\omega_0 L)}{-k\pi}\right)^2 \\ &= \alpha^2 + \sum_{k=1}^{K}\left(\frac{\sin(\pi k\alpha)}{k\pi}\right)^2 + \left(\frac{\sin(-\pi k\alpha)}{-k\pi}\right)^2 \\ &= \alpha^2 + \sum_{k=1}^{K} 2\frac{\sin^2(\pi k\alpha)}{k^2\pi^2}\end{aligned}$$

We need to find the smallest integer K such that:

$$\frac{P_K}{P} \geq 0.95$$

$$\alpha + 2\sum_{k=1}^{K} \frac{\sin^2(\pi k \alpha)}{\alpha k^2 \pi^2} \geq 0.95$$

For $\alpha = 0.15$



By calculating the terms, we find that $N = 2\,K + 1 = 29$ terms are needed to capture at least 95% of the signal's power.